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A THEORY FOR PRESSURE RADIATION FROM OCEAN-BOTTOM EARTHQUAKES

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ABSTRACT

An ocean-bottom earthquake is modelled as motion of a rigid boundary adjacent to a fluid half-space. The resulting water pressure, for a wide class of source motions, is obtained exactly as a convolution integral. The kernel has a physical interpretation as a fundamental solution, and may be obtained explicitly by a Cagniard method. A worked example is given, in which the convolution is carried out, and steps in pressure are found which are approximately equivalent to an extra 200 m in the water column.

Introduction

The source mechanism of earthquakes can be properly understood only if reliable values of source parameters (such as rupture velocity, stress drop, fault geometry) are available. Therefore, a basic aim in seismology is the accurate estimation of earthquake source parameters from seismic data.

However, there has been a lack of theory, not merely for the inversion but also for predicting directly the seismic data which should be obtained from given mechanisms, in all but the simplest cases.

As a contribution to such direct problems, this paper presents a theory for the water pressure variation due to a certain type of motion of the ocean bottom. The actual source description involves parameters of a triggering front, which moves across a restricted area of ocean bottom, and initiates vertical ground motion. The exact pressure solution is obtained in the following section as a spatial and temporal convolution of the ground motion and a fundamental (traveling point source) solution G. In the section on the discussion of the fundamental solution, G is found by the Cagniard-type method due to Gakenheimer (1969) and Gakenheimer and Miklowitz (1969). In the section on a useful worked example, the geometry and parameters of a particular source are used to obtain an estimate of the greatest water pressure which may be expected from a seismic event near the ocean bottom.

The source description used in this paper can easily be extended to model an "earthquake" in an elastic half-space, and it is to be expected that the corresponding fundamental solution, for surface displacements due to a buried traveling point source, can be obtained by Gakenheimer's method (personal communication; June 1970). This paper, then, may be regarded as an introduction to the displacement solution for an earthquake model in the solid earth, which solution it is hoped to present and discuss in a future paper.

Source Description and Water Pressure Solution

1. I first established reasons for taking a very simple model of the ocean and sub-oceanic media, and then give a description of the source. For these media-source models, an exact solution is easily obtained in terms of a certain fundamental solution, to be evaluated below.

2. The application of various source-mechanism theories to observed near-source displacements is the subject of much current seismological research. Housner and

Trifunac (1967) and Aki (1968) conclude from a USCGS accelerogram, written during the Parkfield earthquake (June 28, 1966) at about 20 miles from the epicenter, that the observed ground motion was a displacement pulse of duration about 1.5 sec and amplitude about 25 to 30 cm. Aki modelled this event by a propagating displacement dislocation, of magnitude 60 cm, the dislocation occurring as a step function in time. Brune (1970) has recently discussed an earthquake model described by a tangential stress pulse, leading to a displacement dislocation occurring as a ramp function in time. His model successfully describes near- and far-field displacement time functions and spectra.

In either Aki's or Brune's model, the radiated wave front of stress has a time function which is of higher order than a ramp, and so the greatest fluctuations of stress may be expected to occur just at and after the direct wave arrivals. I draw two conclusions for our problem, in which water pressure corresponds to elastic stress:

If a portion R of deep ocean bottom undergoes the typical near-focus motion of an earthquake, then, (i) the water pressure variations will be greatest in the vicinity of R, and, (ii) at any one place P in such vicinity, the pressure variations are greatest near the arrival times of direct waves from R.

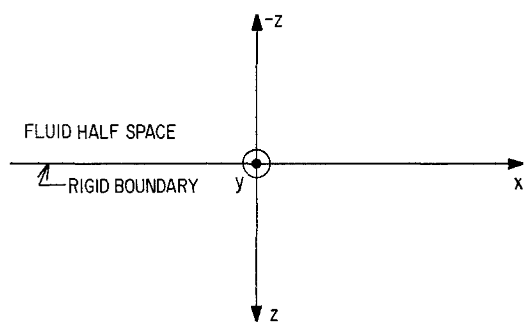

Fig. 1. Coordinate system for the model: x- and y-axes lie in the plane of the ocean bottom, with y increasing normally out of the plane of this illustration.

It is, therefore, reasonable to assume that these pressure variations, which would predominantly consist of short-period energy, are unaffected by (a) the Earth's gravity field, (b) the presence of a free ocean surface, and (c) the tremors propagated, by the initial motion on R, over the ocean bottom.

With these assumptions, our problem reduces to the calculation of that pressure within a *fluid half-space* which results from certain types of motion on the surface of the adjacent *rigid boundary*.

The exact solution I shall develop, for such a medium, can probably be used to solve for an ocean model with a free surface—by working out the expected reflections. Dropping assumption (c) might also be possible, but the details of the fundamental solution would be many times more complicated.

3. Let us now turn our attention to a description of the source, which is some time-dependent system of vertical motion of the (assumed rigid) ocean bottom. The detailed description, given below, is guided by appreciating that our fundamental pressure solution is for a point source, traveling with uniform velocity from some initial position: the finite source is to be synthesized from this basic element.

By choosing cartesian coordinates as in Figure 1, the source is confined to the plane $z = 0$. I shall also suppose that it is confined within some closed region R bounded by two curves $x = f_0(y)$, $x = f_1(y)$ (C_0 and C_1, say), as in Figure 2. That is, C_0 and C_1

together are to form a closed curve (with C_0 on the origin), and R consists of points (x_s, y_s) which lie on lines y = constant, with $f_0(y_s) \leqq x_s \leqq f_1(y_s)$.

For each (x, y) in R, I suppose there is a vertical acceleration (independent of x) which starts at a certain specific time, dependent upon (x, y). To describe this initiation, the concept of a "triggering front" is introduced. Thus, let us suppose such a front starts from the origin at time $t = 0$, and subsequently at (x, y) moves with velocity $V_x(y)$ parallel to the x-axis, and velocity $V_y(y)$ parallel to the y-axis. Require that

(i) $V_x(y)$ is positive
(ii) $V_z(y)$ is nonzero, and has the same sign as y.

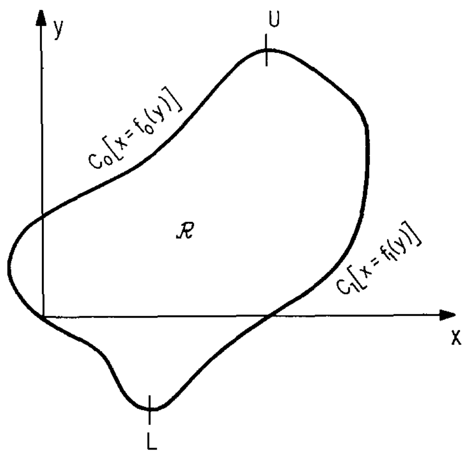

FIG. 2. Parameters for the source region R. Points L and U are linked by the two *curves* C_0 and C_1.

Clearly, this triggering front arrives at (x, y) at time

$$T(x, y) = \frac{x}{V_x(y)} + \int_0^y \frac{d\eta}{V_y(\eta)}$$

$$= \frac{x}{V_x(y)} + \tau(y) \quad \text{(say)} \tag{1}$$

and note, from requirement (ii), that $\tau(y)$ is never negative.

At (x, y) in R, for $t < T$, there is no motion. For $t \geqq T$, the vertical (downward) acceleration assumes a value $a(y, t - T(x, y))$, giving rise to a corresponding change in displacement. In practical cases, the time dependence for this ground motion is such that, as $t \to \infty$, $a = \to 0$ and the total vertical displacement tends to some fixed finite value $Z(y)$ (say). As a simple example for the acceleration function, we could take

$$a(y, t - T(x, y)) = \frac{Z(y)}{\Delta T(y)} [\delta(t - T) - \delta(t - T - \Delta T(y))] \tag{2}$$

(where the δ's are Dirac δ functions) which is suggested by Brune's (1970) work, in which faulting is derived (from the effective available stress) to be a displacement ramp in time. Equation (2) also contains a "locking front", arriving at time $\Delta T(y)$ after the triggering front, and Brune's paper would imply our $Z(y)/\Delta T(y)$ is of the order of 100 cm/sec.

Recalling the relation

$$\rho \ddot{\mathbf{u}} = -\text{grad } p$$

between density ρ, displacement $\mathbf{u}$ (acceleration $\ddot{\mathbf{u}}$), and pressure p, we finally obtain the source description as a boundary condition

$$\left.\frac{\partial p}{\partial z}\right|_{z=0} = -\rho[H(x - f_0(y)) - H(x - f_1(y))]a(y, t - T(x, y))$$

$$\text{for} \quad (x, y) \quad \text{in} \quad R \quad \text{and} \quad t \geqq T$$

$$= 0 \quad \text{otherwise.} \tag{3}$$

(H is the Heaviside unit step function).

4. It is now fairly simple to obtain a solution for pressure in the half-space $z < 0$, assuming p satisfies (a) the wave equation (with velocity α) and (b) the initial conditions that p and $\partial p/\partial t$ are zero at $t = 0$, in $z < 0$.

We take a one-sided Laplace transform of time ($\bar{\ }$), and bilateral Fourier transforms of x and y ($\approx$) to see that

$$\frac{\partial^2}{\partial z^2}\bar{\tilde{\tilde{p}}} = \left(\frac{s^2}{\alpha^2} + k^2 + v^2\right)\bar{\tilde{\tilde{p}}} = n_d^{\,2}\,\bar{\tilde{\tilde{p}}} \quad \text{(say)}$$

where s, k and v are the transform variables corresponding to t, x, y.

It follows that

$$\bar{\tilde{\tilde{p}}} = (\text{some function of } s, k, v)e^{n_d z} \tag{4}$$

in which we make that choice of n_d which has a positive real part (since $\bar{\tilde{\tilde{p}}} \to 0$ as $z \to -\infty$).

The unknown function in (4) may be formed from a transformation of the boundary condition on $z = 0$. From (3),

$$\left.\frac{\partial \bar{p}}{\partial z}\right|_{z=0} = -\rho[H(x - f_0(y)) - H(x - f_1(y))]e^{-sT}\bar{a}(y, s).$$

Taking the x-transform, and using formula (1) for $T(x, y)$ together with the briefer notation $c = V_x(y)$,

$$\left.\frac{\partial \tilde{\bar{p}}}{\partial z}\right|_{z=0} = -\rho e^{-s\tau(y)}\frac{c}{s + ick}[e^{-((s/c)+ik)f_0(y)} - e^{-((s/c)+ik)f_1(y)}]\bar{a}(y, s).$$

Taking the y-transform, and using (1) and (4), it then follows that

$$\bar{\tilde{\tilde{p}}}(k, v, z, s) = \int_{-\infty}^{\infty}\left\{\left[\frac{-\rho c(y')e^{n_d z}}{n_d(s + ic(y')k)}\right] \cdot e^{-ivy'}\bar{a}(y', s)\right.$$

$$\left.\cdot[\exp - (ikf_0(y') + sT(f_0(y'), y')) - \exp - (ikf_1(y') + sT(f_1(y'), y'))]\right\} dy'. \tag{5}$$

Note (i). In (5) I have used y' as the (dummy) integration variable, merely to assist the recognition of a spatial convolution below.

Note (ii). The disturbance is initiated at $x = 0$, $y = 0$, and thus $T(f_1(y), y) \geqq T(f_0(y), y) \geqq 0$ [from (1)]. The real exponents in (5) are, therefore, never positive, and there is no difficulty in showing convergence of the integrand, for non-negative real s.

Note (iii). In practical applications, $\bar{a}(y, s) = 0$ for y greater than some $y(U)$, say (see Figure 2), or for y less than some $y(L)$. Then $y(U)$ and $y(L)$ appear as upper and lower limits of the integration in (5).

Suppose now that we can calculate the function $G(\mathbf{x}, t; c(y'))$ for which

$$\bar{\bar{\bar{G}}}(k, v, z, s; c(y')) = \frac{-\rho c(y')e^{n_d z}}{n_d(s + ic(y')k)}. \tag{6}$$

(Just this calculation is the subject of the section below). Then it is straightforward to recognize the pressure solution as

$$p(x, y, z, t) = \int_0^t dt' \int_{-\infty}^{\infty} a(y', t - t').$$

$$\{G[x - f_0(y'), y - y', z, t' - T(f_0(y'), y'); c(y')]H(t' - T(f_0(y'), y'))$$

$$-G[x - f_1(y'), y - y', z, t' - T(f_1(y'), y'); c(y')]\, H(t' - T(f_1(y'), y'))\}\, dy'. \tag{7}$$

This formula is basically a spatial and temporal convolution of the source acceleration and two fundamental solutions G: the first G-function corresponds to a point source traveling across R from an initial point on $C_0[x = f_0(y)$: see Figure 2], and the second G-function essentially "turns off" the first, when $C_1[x = f_1(y)]$ is reached. The convolution over y' is required by the finiteness of the source, and the convolution over t' to convert the time function implicit in G (see below) into the actual source-time function $a(y, t - T)$ (for acceleration).

In using (7), recall that $c(y') = V_x(y')$ is the velocity component, at y', of the triggering front, parallel to the x-axis; T is the time-delay function defined in (1); and, if R is restricted to lie between $y(L)$ and $y(U)$, the y' integration is also taken between these limits.

I shall show below that the bracket $\{\ \}$ of (7) is made up of (a), a wave system emanating from the boundary of R, and (b), a stronger wave system (but reaching a limited region of $z < 0$) emanating from each point of the triggering front, as it crosses R. Also, in the section on a useful worked example, I carry out the integrations of (7), for the wave system (b), for a particular example of source geometry, and for the acceleration function (2).

Discussion of the Fundamental Solution $G(\mathbf{x}, t; c)$

We should note that G, introduced above, following (5), can be regarded as the "pressure" field due to the vertical "acceleration" boundary condition

$$a_z\,|_{z=0} = \delta(t - x/c)\delta(y)H(x) \tag{8}$$

which is a point source, starting at the origin and moving with speed c along the positive x-axis. That is, (8) gives

$$\tilde{\tilde{G}}(k, v, z, s; c) = \frac{-\rho c e^{n_d z}}{n_d(s + ick)}$$

which is equation (6) again. (G would actually be a pressure if an appropriately-dimensioned constant were included in the right-hand side of (8), to yield an acceleration.)

Note too, that the final argument of G is a source velocity, and is independent of $\mathbf{x}$. This argument is labeled c in the following discussion of G, and for evaluation of the formulas (1) and (7), would be assigned by $c = c(y') = V_x(y')$. I shall consider only the case $c > \alpha$, i.e., a supersonic source, because $\alpha \sim 1.5$ km/sec in water, and this is almost certainly lower than the rupture velocity of earthquake faulting.

It is most fortunate that G can be recognized as the pressure due to source (8), because a related problem of a normal point load traveling from some initial position on the surface of an elastic half-space has recently been exactly solved by Gakenheimer (1969) and Gakenheimer and Miklowitz (1969). This related problem, with its P and S velocities, head waves, surface waves, and subsonic, transonic and supersonic source speeds, may well have several important applications in seismology, but it is far more complicated than our supersonic problem of a fluid half-space and rigid boundary. My derivation of G is given in Appendix A. It closely follows Gakenheimer and Miklowitz's (1969) notation, and is a useful tutorial example of their elegant method. The solution is

$$G(\mathbf{x}, t; c) = A(\mathbf{x}, t) + B(\mathbf{x}, t)$$

where, from (A5 and (A7),

$$A(\mathbf{x}, t) = \frac{-\rho\alpha}{\pi^2 R} H(t - R/\alpha) \int_0^{T_d} Re\left[\frac{L + iM}{(L + iM)^2 + w^2 \sin^2 \theta}\right] \frac{dw}{(T_d^2 - w^2)^{1/2}}$$

$$L = \frac{\alpha}{c}\left(1 - \frac{ctx}{R^2}\right), \qquad R = (x^2 + y^2 + z^2)^{1/2}, \tag{9}$$

$$M = R^{-1}|z| \cos\theta(T_d^2 - w^2)^{1/2}, \qquad \theta = \tan^{-1}(y/x),$$

$$T_d = (t^2\alpha^2 R^{-2} - 1)^{1/2}, \text{ and}$$

$$B(\mathbf{x}, t) = \frac{-\rho c}{\pi} \frac{H(t - t_c)H(t - R^2/cx)H(x)}{[(ct - x)^2 - (ct_c - x)^2]^{1/2}} \tag{10}$$

where

$$t_c = [(c^2\alpha^{-2} - 1)^{1/2}(y^2 + z^2)^{1/2} + x]/c.$$

Several interesting properties of G may be derived from (9) and (10):

(i) $A(\mathbf{x}, t)$ is a pressure disturbance initiated at $\mathbf{x} = \mathbf{0}$, $t = 0$, and confined behind the hemispherical wave front $t = R/\alpha$ for $t > 0$. It is easy to show

$$A(\mathbf{x}, t) = -\frac{\alpha\rho H(t - R/\alpha)}{2\pi(x - R\alpha/c)} [1 + 0(t - R/\alpha)] \tag{11}$$

as $t \to R/\alpha$, i.e. that the wave front is a step in the pressure.

(ii) $B(\mathbf{x}, t)$ is a pressure disturbance confined behind the conical wave front, $t = t_c$, set up in the fluid by the moving (supersonic) point source. For $\mathbf{x}$ in $z < 0$ such that this wave front is the first arrival (i.e., for $\mathbf{x}$ in region I of Figure 5), it is easy to show

$$B(\mathbf{x}, t) = \frac{-\rho}{\pi} \frac{H(t - t_c)[1 + 0(t - t_c)]}{2^{1/2}(t_c - x/c)^{1/2}(t - t_c)^{1/2}} \tag{12}$$

as $t \to t_c$, i.e., that the wave front is a (negative) integrable singularity in the pressure.

(iii) A complication arises in evaluating (9) and (10) for $\mathbf{x}$ in region II of Figure 5. For, the term $B(\mathbf{x}, t)$ is, in this region, discontinuous across the time $t = R^2/cx$ which may be thought of as part of the surface of a sphere having the origin and the source point as a diameter (see Figure 5), and so B has a spherical wave front in region II. Gakenheimer and Miklowitz (1969) remark that this surface is not expected to be a wave front of $G = A + B$, and in Appendix B, I prove that indeed $A(\mathbf{x}, t)$ does carry a matching discontinuity. The proof gives as a corollary the special algebraic value

$$G(\mathbf{x}, R^2/cx; c) = -\frac{\rho\alpha x}{2\pi(y^2 + z^2)^{1/2}(R^2\alpha^2c^{-2} - x^2)^{1/2}}$$

on the surface $t = R^2/cx$, within region II.

(iv) If the relation $G = A + B$ is used in the convolution formula (7), we see that the pressure due to the finite source R can be decomposed into (a) a system of disturbances, confined behind the envelope of hemispherical wave fronts, which emanate from each point on the edge of the source region, and (b) a system of disturbances, confined behind the envelope of conical wave fronts, which emanate from each point on the triggering front, as it moves across R. This latter system is "turned off" in space by a canceling system, [the second function G of (7)], outside the physical source region: the cancellation is apparent from formulas (7) and (10), because the expression $t - t_c$ of (10) works out to the same value if arguments either from the first G or the second G in (7) are used.

Many properties of the total solution are illustrated by the following example, which permits evaluation of the convolution integrals in (7).

A Useful Worked Example

Let us consider a source of type (2), with ΔT and Z independent of y. Also take $V_y(y) = \infty$ (so $\tau(y) = 0$), and let R be the strip $0 \leqq x \leqq X$.

That is, the source initiates at time $t = 0$ as a rising infinite line ($x = 0$) of ocean bottom, moving up with velocity $-Z/\Delta T$ for ΔT seconds, and then stopping. This line source moves parallel to the x-axis with velocity c ($c > \alpha$, the sound speed in water), acting as a triggering front for vertical motion, and stops at the line $x = X$. In Figure 3, the wave fronts for this simple problem are shown at some time $t > \Delta T + X/c$ (i.e., after they have all had time to develop). Wave fronts 1 and 1′ are the envelopes of conical waves for (respectively) the triggering and locking fronts on R. Cylindrical wave fronts 2 and 2′ are due to initiation of triggering and locking at $x = 0$, i.e., from $G = A$ for the first G function in (7), and 3 and 3′ from the termination of triggering and locking at $x = X$, i.e., from initiation of $G = A$ for the second G function of (7). I now find the exact pressure changes due to 1 and 1′ (comparing the result with a direct exact method, for this source), and also find the jumps in pressure across fronts 2 and 2′, 3 and 3′:

Define V_I to be the region of $z < 0$ such that wave front 1 is the first arrival (see Figure 3). Then pressure p_1 associated with this wave front is obtained from (7) by substituting the trigger $a(t) = Z\delta(t)/\Delta T$, and the conical component B from (10), giving

$$p_1(\mathbf{x}, t) = -\int_{-\infty}^{\infty} \frac{\rho c Z}{\pi \Delta T} \frac{H(t - t_c(y'))\, dy'}{[(ct - x)^2 - (ct_c(y') - x)^2]^{1/2}} \tag{13}$$

where

$$t_c(y') = c^{-1}\{(c^2\alpha^{-2} - 1)^{1/2}[(y - y')^2 + z^2]^{1/2} + x\}.$$

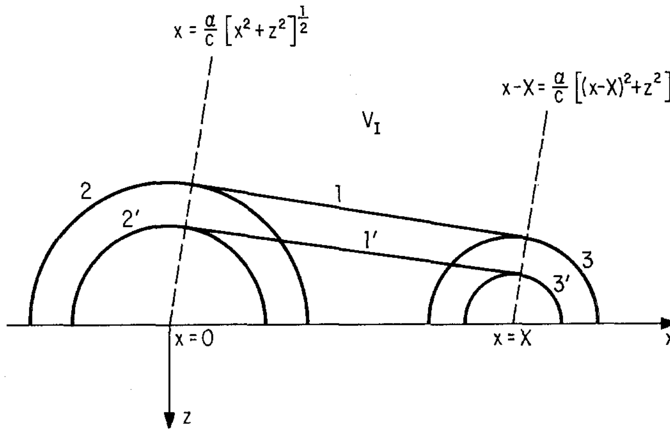

FIG. 3. A section (y = constant) through the cylindrical wave fronts due to a source in the strip $0 \leqq x \leqq X$, $z = 0$.

Clearly, $t \geqq t_c(y)$ for there to be a nonzero p_1. Defining $D(y') = ct_c(y') - x$ as the variable of integration, the minimum of D is $D(y)$, and so

$$p_1(\mathbf{x}, t) = \frac{-\rho c Z}{\pi \Delta T} H(t - t_c(y))2 \int_{D(y)}^{ct-x} \left|\frac{dD}{dy'}\right|^{-1} \frac{dD}{[(ct - x)^2 - D^2]^{1/2}}.$$

The integral here is just $(c^2\alpha^{-2} - 1)^{-1/2}\pi/2$, and we find the simple result

$$p_1(\mathbf{x}, t) = \frac{-\rho\alpha Z H[t - x/c - |z|(1 - \alpha^2/c^2)^{1/2}/\alpha]}{\Delta T(1 - \alpha^2/c^2)^{1/2}} \tag{14}$$

for the disturbance 1, valid everywhere in V_I.

Using the locking function $a(t) = -Z\delta(t - \Delta T)/\Delta T$, a formula for $p_{1'}(\mathbf{x}, t)$ can similarly be worked out, and in fact $p_{1'}(\mathbf{x}, t) = -p_1(\mathbf{x}, t - \Delta T)$. The exact pressure record at a point in V_I, due to these two pressure contributions, is thus a step jump from zero, which is held constant at $-\rho\alpha Z(1 - \alpha^2/c^2)^{-1/2}/\Delta T$ for a period ΔT, and then steps back to zero. Taking values $\rho = 1$, $\alpha = 1.5 \times 10^5$, $c = 3 \times 10^5$ (all cgs units), and Brune's (1970) suggestion $Z/\Delta T = -100$ cm/sec (for an upward source velocity), the pressure jump would be $3^{1/2} \times 10^7$ dynes/cm^2: interpreting this in the Earth's gravity field, the jump is equivalent to about another 180 m in the water column.

A reassuring check on our general formulas (7) and (10) is provided by noting that the derived result (14) can be obtained directly for this particular source. For, the pressure p_1 in region V_I is equivalent to that due to a steadily moving source (which has traveled from $x = -\infty$), and so

$$\frac{\partial}{\partial z} p_1(x, y, 0, t) = -\rho Z \delta(t - x/c)/\Delta T \tag{15}$$

[cf. equation (3)]. The wave front 1 travels along the x-axis with speed c, and, hence, along the negative z-axis with speed $(1 - \alpha^2/c^2)^{-1/2}\alpha$, so by causality we have $p_1(\mathbf{x}, t) = P(\mathbf{x}, t)H(t - x/c + z(1 - \alpha^2/c^2)^{1/2}/\alpha)$ for some P. Substituting this form into (15), and requiring that p_1 satisfy the wave equation,

$$P = \text{constant} = -\rho\alpha Z(1 - \alpha^2/c^2)^{-1/2}/\Delta T$$

can be claimed by inspection, and (14) is verified.

The pressure jumps across the cylindrical wave fronts 2, 2′, 3, 3′ (see Figure 3) may be expected to be smaller than those across 1, 1′, because of the correspondence with wave fronts of the fundamental source, [equations (11) and (12), which have different orders of discontinuity]. It is not difficult to obtain the pressure p_2 across wave front 2, [by substituting (11) into (7), together with the δ function for $a(t)$], in the form

$$p_2(\mathbf{x}, t) = \frac{\rho Z\alpha[2R_0(\alpha t - R_0)]^{1/2}}{\pi\Delta T(x - R_0\alpha/c)} H(t - R_0/\alpha) \quad \text{as} \quad t \to R_0/\alpha = (x^2 + z^2)^{1/2}/\alpha. \tag{16}$$

It follows that p_2 is continuous across wave front 2, but then increases in magnitude fairly rapidly. If (16) is accurate for $t - R_0/\alpha = 1$ sec, then p_2 could attain the magnitude of p_1 within 1 or 2 km of the initial source rupture. It could also be large near $x = \alpha R/c$ (shown as a *dashed line* in Figure 3), since there the factor L^{-1} of (10) would be large, near the time $t = R/\alpha$.

The wave fronts 2′, 3 and 3′, shown in Figure 3, are of the same type as that expressed by formula (16): the discontinuities they carry can be evaluated by inserting in (16) a minus sign for 2′ and 3; by replacing the quantities x and R_0 by $x - X$ and $[(x - X)^2 + z^2]^{1/2}$, for 3 and 3′; and by using arrival times $t = (x^2 + z^2)^{1/2}/\alpha + \Delta T$, $t = [(x - X)^2 + z^2]^{1/2}/\alpha + X/c$ and $t = [(x - X)^2 + z^2]/\alpha + \Delta T + X/c$ for 2′, 3 and 3′ (respectively).

Finally, it may be noted that, although our wave front analysis has used an infinite (moving) line source, formulas (14) and (16) would also be exact for a finite line source, providing the receiver at $\mathbf{x}$ is situated between two planes each of which lies normally on an end of the line. [This follows, because the wave front is synthesized from waves emanating from a small neighborhood of the source surrounding the nearest point of the source to x: see the evaluation of (14) from (13).] So solutions (14) and (16) have a practical relevance as the biggest pressure changes which may be expected from finite sources, assuming the time function (2).

Conclusions

This paper has modelled the water pressure, due to an ocean-bottom earthquake, by that pressure field within a fluid half-space which results from motion on the surface of the adjacent rigid boundary. For this model, the exact pressure solution is found for those source motions which can be synthesized from a certain fundamental source, namely, a vertical acceleration which initially occurs (as a δ function in time) at one point of the ocean bottom, and which subsequently travels with constant velocity.

Such a fundamental source is capable of synthesizing many different useful types of finite source (see section on source description, paragraph 3); a formula for the pressure synthesis [see equation (7)] is seen to be a spatial and temporal convolution of the fundamental solution, G, with the source region acceleration. (Such convolutions are to be expected, since they represent the most general linear superposition of fundamental solutions.)

The solution G is worked out in Appendix A, using the method of Gakenheimer and Miklowitz (1969), and is expressed as the sum of (a) a pressure disturbance spreading behind a hemispherical wave front from the initial point of source motion, and (b) a pressure disturbance spreading behind a conical wave front with apex at the traveling source.

The convolution integrals are worked out, in the section on a useful worked example, for particular source geometry using Brune's (1970) suggestion for the time function as a step in velocity. For this source, the water pressure would change by an amount equivalent to a difference of about 200 m in the water column.

The general method of this paper may be expected to apply in seismological problems of faulting in the solid earth. From equations (14) and (16) we can make the interesting deduction that the Fourier time transform of displacement would have components with high-frequency amplitude behaviors like $(\text{frequency})^{-2}$ from (14) and $(\text{frequency})^{-5/2}$ from (16).

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References

Aki, K. (1968). Seismic displacements near a fault, *J. Geophys. Res.* 73, 5359–5376.

Brune, J. N. (1970). Tectonic stress and the spectra of seismic shear waves from earthquakes, *J. Geophys. Res.* 75, 4997–5009.

Gakenheimer, D. C. (1969). *Ph.D. Thesis,* California Institute of Technology.

Gakenheimer, D. C. and J. Miklowitz (1969). Transient excitation of an elastic half space by a point load traveling on the surface, *J. Appl. Mech.* 36, 505–515.

Housner, G. W. and M. D. Trifunac (1967). Analysis of accelerograms—Parkfield Earthquake, *Bull. Seism. Soc. Am.* 57, 1193–1220.

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Appendix A

The Inversion of $\bar{\bar{\bar{G}}}(k, v, z, s; c)$

Inverting the x, y transforms of (b), find,

$$\bar{G}(\mathbf{x}, s; c) = \frac{-1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\rho c e^{n_d z + i(kx + vy)}}{n_d(s + ick)} \, dk \, dv. \tag{A1}$$

Making the de Hoop transformation, we replace (k, v) in (A1) by

$$k = \frac{s}{\alpha}(q \cos\theta - w \sin\theta), \qquad v = \frac{s}{\alpha}(q \sin\theta + w \cos\theta)$$

where $\theta = \tan^{-1}(y/x)$ and $r = (x^2 + y^2)^{1/2}$. After separating the resulting integrand into components even and odd in w, we obtain

$$\bar{G}(\mathbf{x}, s; c) = \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty K(q, w, \theta) e^{(s/\alpha)(mz + iqr)}\, dq\, dw \qquad \text{(A2)}$$

in which

$$K(q, w, \theta) = \frac{-\rho}{\pi^2} \frac{iq\cos\theta + \gamma}{[(iq\cos\theta + \gamma)^2 + w^2 \sin^2\theta]} \frac{1}{m}$$

$$\gamma = \frac{\alpha}{c} = \frac{\text{sound speed in water}}{\text{source speed}} < 1 \text{ for a supersonic source}$$

$$m = \alpha n_d/s = (q^2 + w^2 + 1)^{1/2}$$

and, since $z < 0$, we require m to have a positive real part. (A2) is symmetric in θ, so the range $0 \leqq \theta \leqq \pi$ is used for the inversion procedure.

The singularities in the integrand of (A2) are (a) branch points at $q = Q^\pm \equiv \pm i(w^2 + 1)$, which points also are poles, and (b) simple poles at $q = Q_c^\pm \equiv (\pm w \sin\theta + i\gamma)/\cos\theta$.

We next attempt to convert $\bar{G}$ into the Laplace transform of a recognizable time function by using the exponent factor $-(mz + iqr)/\alpha$ as a single variable, labelled t, and seeing what curve in the complex q-plane is equivalent to varying real positive values of t. Solving for q, we find the paths,

$$q = q^\pm \equiv [itr \mp z(t^2 - t_w^2)^{1/2}]\alpha/R^2 \quad \text{for} \quad t \geqq t_w \qquad \text{(A3)}$$

where

$$R = (r^2 + z^2)^{1/2} = (x^2 + y^2 + z^2)^{1/2}, \quad \text{and} \quad t_w = (w^2 + 1)^{1/2} R/\alpha.$$

Equation (A3) defines one branch of a hyperbola, shown in Figure 4, together with the singularities of the integrand in (A2). The vertex lies between Q^+ and the origin (since $r < R$), and so paths $q^\pm$ do not cross the branch cut. The additional contribution of integrals along large arcs C_I, C_{II} (see Figure 4), to the real q-axis integration (A2), is vanishingly small. So,

$$\begin{aligned}\bar{G}(\mathbf{x}, s; c) = &\frac{1}{2}\int_0^\infty \int_\infty^{t_w} K(q^-(t), w, \theta) \frac{dq^-}{dt} e^{-st}\, dt\, dw \\ &+ \frac{1}{2}\int_0^\infty \int_{t_w}^\infty K(q^+(t), w, \theta) \frac{dq^+}{dt} e^{-st}\, dt\, dw \\ &+ \frac{1}{2}\int_0^\infty \{\text{the sum of residues of } K \text{ at } q = Q_c^\pm(w), \text{ for } (w, \mathbf{x}) \\ &\quad \text{such that these poles lie between } q^\pm \text{ and the real } q\text{-axis}\}\, dw \qquad \text{(A4)}\end{aligned}$$

Denoting the first two terms of (A4) by $\bar{A}(\mathbf{x}, s)$ and the last by $\bar{B}(\mathbf{x}, s)$, our next step is the evaluation of $A(\mathbf{x}, t)$.

It can be shown that $[-K(q^-, w, \theta)\, dq^-/dt]$ and $[K(q^+, w, \theta)\, dq^+/dt]$ are complex conjugate functions (of real t). So

$$\bar{A}(\mathbf{x}, s) = \int_0^\infty \int_{t_w}^\infty Re\left[K(q^+, w, \theta)\, \frac{dq^+}{dt}\right] e^{-st}\, dt\, dw.$$

After interchanging this order of integration, the inverse Laplace transform is recognized as

$$A(\mathbf{x}, t) = H(t - R/\alpha) \int_0^{((t^2\alpha^2/R^2)-1)^{1/2}} Re\left[K(q^+, w, \theta)\, \frac{dq^+}{dt}\right] dw \qquad \text{(A5)}$$

which is a pressure wave initiated at $x = 0$, and subsequently is a disturbance behind the hemispherically spreading wave front $t = R/\alpha$. The formula (9), given in the text, is obtained from (A5) by inserting the definitions of K and q^+.

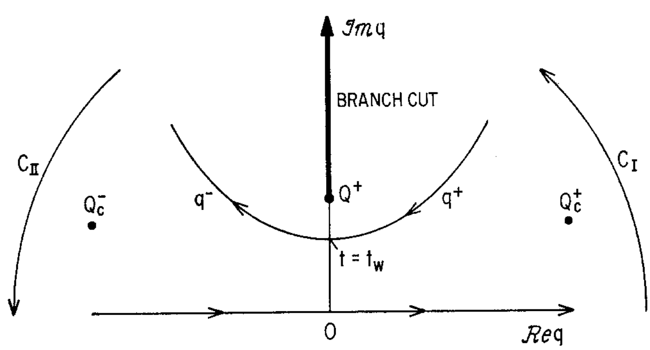

FIG. 4. Singularities of the integrand for $\bar{G}$ [equation (A2)] and integration paths in the q-plane w is real.

The method for inversion of the last term in equation (A4) [$\bar{B}(\mathbf{x}, s)$] is the single most important feature of Gakenheimer and Miklowitz's (1969) paper.

Criteria are first established for $(w, \mathbf{x})$ to be such that poles $Q_c^\pm$ lie between curves $q^\pm$ and the real q-axis (see Figure 4).

Division of $z < 0$ into three regions is depicted in Figure 5:

Region I: $x > 0$, $x > \gamma R$. The poles are included for all w, $0 \leqq w < \infty$ [since $Im\, Q_c^\pm(w)$ is positive, but less than the vertex height of hyperbola $q^\pm$].

Region II: $x > 0$, $x < \gamma R$. A little algebra shows that the poles are included for w in the range $w_0 < w < \infty$, where $w_0 = |z|\,(\gamma^2 R^2 - x^2)^{+1/2}(y^2 + z^2)^{-1/2} r^{-1}$, but lie on $q^\pm$ for $w = w_0$, and above $q^\pm$ for $0 \leqq w < w_0$.

Region III: $x < 0$. The poles are not included, since they lie below the real q-axis.

It follows immediately that $\bar{B}(\mathbf{x}, s)$, and $B(\mathbf{x}, t)$, are zero in region III.

In regions I and II the integrand { } of (A4), for $\bar{B}$, can be found by simple residue theory, and gives

$$\bar{B}(\mathbf{x}, s) = \frac{-\rho \sec\theta}{\pi} \int^\infty Re\left(\left.\frac{e^{s/\alpha(mz+iqr)}}{m}\right|_{q=Q_c^+}\right) dw \qquad \text{(A6)}$$

in which the lower limit of integration is 0 for region I, and w_0 for region II. The integrand singularities are, from the definition of $m(q, w)$, merely branch points (and poles) at $w = S^{\pm} \equiv -i\gamma \sin\theta \pm i(1-\gamma^2)^{1/2} \cos\theta$, and a Cagniard-de Hoop path in the complex w-plane is now given (see Figure 6) by

$$t = -(mz + iqr)/\alpha \Big|_{q = Q_c^+(w)}.$$

That is,

$$w = w^{\pm} \equiv -i\gamma \sin\theta + \gamma \cos\theta[iy(ct-x) \mp zA]/n^2 \tag{A7}$$

for $t \geqq t_c$, where $n = (y^2 + z^2)^{1/2}$, $A = [(ct-x)^2 - (ct_c - x)^2]^{1/2}$, and $t_c = [(1-\gamma^2)^{1/2} \cdot n + \gamma x]/\alpha$ is the arrival time of a conical wave which trails behind the source (see Figure 5).

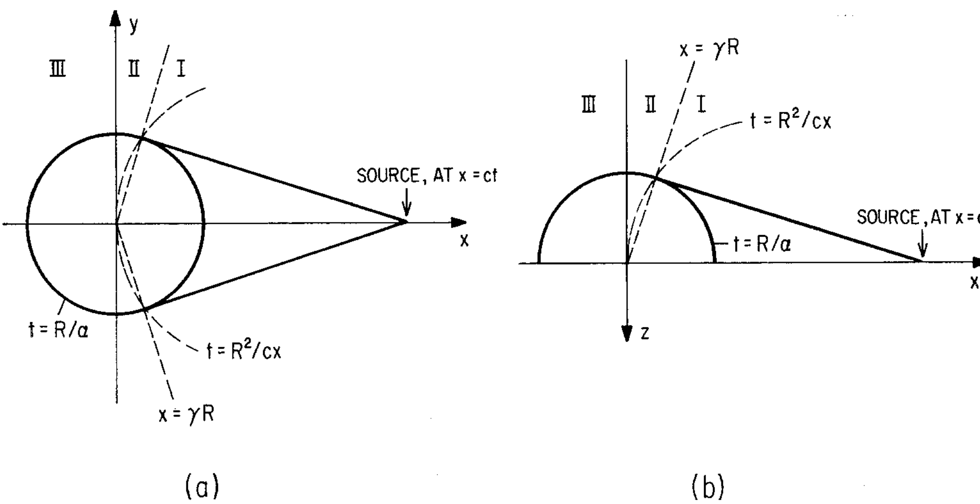

FIG. 5. (a) Wave fronts in plane $z = 0$, at time $t > 0$. (b) Wave fronts in plane $y = 0$, at time $t > 0$.

Equation (A7) defines one branch of a hyperbola, with vertex $w = -i\gamma \sin\theta + i(1-\gamma^2)^{1/2} y \cos\theta/n$, and, since the integration (A6) is contained on the real positive w-axis, only w^+ is needed. This path is shown in Figure 6. For region I, w^+ intersects the imaginary axis above the real w-axis (since $x > \gamma R$) and below S^+ (since $y < n$). For region II, $w^+(t)$ intersects the real w-axis at $w = w_0$, where $t = R^2/cx$. The integral (A6) taken along C_0 is zero, because its real part is zero, and taken along C_1, is vanishingly small. So in regions I and II,

$$\bar{B}(\mathbf{x}, s) = \frac{-\rho \sec\theta}{\pi} \int_{\mathrm{Min}(t_c, R^2/cx)}^{\infty} Re\left[\frac{1}{m(Q_c^+(w^+(t)))} \frac{dw^+}{dt}\right] e^{-st}\, dt$$

and, clearly,

$$B(\mathbf{x}, t) = \frac{-\rho \sec\theta}{\pi} H(t - t_c) H\left(t - \frac{R^2}{cx}\right) Re\left[\frac{1}{m(t)} \frac{dw^+}{dt}\right]. \tag{A8}$$

The simple formula (10), given in the text, is obtained from (A8) by inserting the definitions of w^+, Q_c^+, m, and t_c. Note that $B = 0$ in region III, which fact can be incorporated into (A8) by including a factor $H(x)$. Finally, G is obtained from (A5) and (A8) using $G = A + B$.

Appendix B

Continuity of $G(\mathbf{x}, t; c)$ as $t \to R^2/cx$ in Region II

It follows from equation (10) that

$$B(\mathbf{x}, t) \to -\frac{\rho\alpha x}{\pi n} (\gamma^2 R^2 - x^2)^{-1/2} H(t - R^2/cx) \tag{B1}$$

as $t \to R^2/cx$, for $\mathbf{x}$ in region II (see Figure 5). In a similar problem involving a solid elastic half-space, Gakenheimer (1969) has shown for the special case $z = 0$ that $A(\mathbf{x}, t)$ has a canceling discontinuity at $t = R^2/cx$, and hence that G is continuous across this time. I prove this result here for all $\mathbf{x}$ in region II.

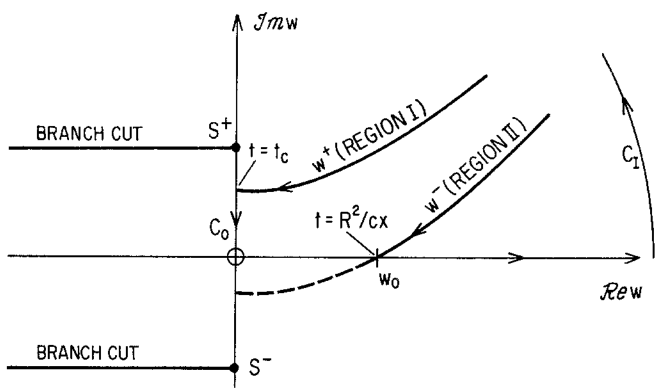

Fig. 6. (b) Singularities of the integrand for $\bar{B}$ [equation (A6)] and integration paths in the w-plane. [Note added in proof: the curve through w_0 should also be labeled w^+.]

It follows from (9) that the integrand for $A(\mathbf{x}, t)$ has a pole at $w = w_p$ (say) (a zero of $w \sin \theta - M + iL$), which migrates across the path of integration as t increases through R^2/cx. ($\mathbf{x}$ is taken within region II for the discussion throughout this Appendix.) Furthermore, it may be shown that the integrand in (9) contributes to $A(\mathbf{x}, t)$ only from the neighborhood of this pole, since the remainder of the path contributes only $0(t - R^2/cx)$. To calculate the limit of A as $t \to R^2/cx$, it is, therefore, sufficient to approximate the integration by the method

$$A(\mathbf{x}, t) \sim ReN \int_{Rew_p - \epsilon}^{Rew_p + \epsilon} \frac{dw}{w - w_p} \sim ReN \int_{-\infty}^{\infty} \frac{dw}{w - w_p}$$

$$= ReN \int_{-\infty}^{\infty} \frac{dw}{w - iIm(w_p)} = ReNi\pi \operatorname{sgn}[Im(w_p)] \tag{B2}$$

as $t \to R^2/cx$ (where sgn $a \equiv \pm 1$ according as $a > 0$ or $a < 0$). It thus remains merely to find N and $Im(w_p)$.

The definition of L and $M(w)$ [equation (9)] leads to

$$w_p = +\left[w_0^2 - \frac{2\alpha z^2 R^2 L}{cr^2 n^2} + \frac{z^2 R^2 L^2}{r^2 n^2}\right]^{1/2} - \frac{iyR^2 L}{rn^2} \tag{B3}$$

where $w_0 = (R^2\alpha^2c^{-2} - x^2)^{1/2} | z |/rn$. As $t \to R^2/cx$, we have $L \to 0$ and $w_p \to w_0$. So, in the neighborhood of w_p (and hence of w_0), we obtain from Taylor expansions the formulas

$$(L + iM)^2 + w^2 \sin^2 \theta \sim 2w_0n^2(w - w_p)/R^2$$

$$(L + iM)/(T_d^2 - w^2)^{1/2} \sim i\alpha | z | x/rR^2.$$

From equation (9) we then see that the N of (B2) is given by

$$N = \frac{-i\rho\alpha |z| x}{2\pi^2 r w_0 n^2} \tag{B4}$$

and also $\operatorname{sgn} Im\, w_p = -\operatorname{sgn} L = \operatorname{sgn}(t - R^2/cx)$. (B5)

From (B2), (B4), (B5) and the definition of w_0, we have

$$A(\mathbf{x}, t) \to \frac{\rho\alpha x}{2\pi n(\gamma^2R^2 - x^2)^{1/2}} \operatorname{sgn}(t - R^2/cx) \tag{B6}$$

as $t \to R^2/cx$, and the required continuity of $G = A + B$ follows immediately from (B1) and (B6).